

# Renormalization Induced Quantum Small-World Networks

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Quantum networks are critical to quantum communication and distributed quantum computing. Here we propose a small-world model of scalable quantum repeater network (QRN), where small-world is a key concept rooted in complex networks. We demonstrate that QRN with scale-free-fractal structure can be successively enlarged into large-scale ones, with topological structure and hierarchy of measurements consistently preserved by performing renormalization transformation. In fact, renormalization here serves as an organizing principle that shapes the topology and governs distribution of long-range entangled links over QRN. Furthermore, we prove that iterative renormalization is capable of triggering fractal to small-world transition, and eventually a hierarchical quantum small-world network is achieved. We expect it could shed light on the design of QRN.

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The goal of quantum networks is to facilitate the propagation of entanglements and transmission of quantum states over networks. Quantum networks are composed of quantum nodes, where qubits are stored and manipulated locally [1]. These nodes are connected by entanglements. As real as energy, entanglement is essential ingredient for quantum information and lies at the heart of quantum physics. Several entanglement-assisted quantum communication schemes were devised [2], such as teleportation that can faithfully teleport arbitrary unknown quantum states, and quantum cryptography (Ekert91 protocol) which enables truly secure communication. However, due to the fatal photon loss and state decoherence, the state fidelity (overlap with a singlet) decreases exponentially with the channel length, which severely limits the distance of quantum communication, and eventually disables long distance communication. Quantum repeater protocols (QRP) [2–5] are one of most promising solutions toward long-distance quantum communication. Thus, it has a realistic significance to investigate QRN, yet a practical design of which is absent. The basic idea of quantum repeaters is elucidated in Fig. 1(a) and (b).

Motivated by the significant achievements on complex networks in the past decade, we propose a small-world model of scalable QRN incorporating two active fields: quantum information [2] and complex networks [7]. The crossover study has provided us new phenomena and fresh perspective. For example, it was found in quantum random networks that the exponent of critical probability at which special subgraphs emerge unexpectedly collapses onto a smaller optimal value, which remarkably differs from classical counterpart [6]. As quantum networks grow continuously, the topological structure, which determines dynamic functions, will become increasingly relevant. In this paper, we aim at exploiting the principle of quantum repeaters to construct quantum network with special topological structure that fulfills practical requirements.

Indeed, it's an open problem that we can't circumvent to conceive of the possible topological structure of future quantum networks. Before formally discussing this question, it's instructive to refer to diverse real networks ranging from biological, technological and social networks. Despite the great

diversity, these real networks share general features, all of which are supported by enormous empirical data and newly developed theories. Two fundamental characteristics deserve concern. Firstly, most of them are scale-free networks. A network is scale-free [7, 8] if its degree distribution  $P(k)$ , i.e., the probability to find a node with  $k$  links follows a power law  $P(k) \sim k^{-\gamma}$ . Secondly, small-world is a generic property of both natural and artificial networks [7, 9]. It's typically characterized by the slow increase of average diameter (average distance [10] between any pair of nodes)  $\bar{\ell}$ , with total number of nodes  $N$ :

$$\bar{\ell} \sim \ln N, \quad (1)$$

Although substantial advances have been achieved for realizing 1D quantum repeaters, rules concerning topological structure which 2D QRN is subject to remain a gap. For large-scale QRN, quantum nodes are connected by long-range entangled links (*shortcuts*). However, direct connection is not available, because of the aforementioned limitation. There is no doubt that this imposes strong constraints on construction of QRN, which we call *local attachment*: a node prefers to be spatially entangled with neighboring nodes, rather than remote nodes. Hence, these quantum nodes are instead connected via repeater nodes.

It's of fundamental importance to clarify whether the topology of quantum networks are completely exceptions, compared with that of real networks. We conjecture that QRN are most likely scale-free-fractal networks observed at a length-scale. We believe it is reasonable, in that it's strongly supported by three facts. It has been experimentally verified that entangled photon pairs can be transmitted through commercial telecom fiber. Other than local operations, classical communication is indispensable between quantum nodes. Therefore, to some extent, quantum networks are embedded in classical communication networks. Whereas both phone call networks and Internet are scale-free networks [7]. The local attachment style implies strong repulsion between hubs. That is, nodes with large degrees are connected via those of small degrees, leading to fractal architecture [11] in large-scale space. Many biological networks are found to be scale-free-fractal

networks[12, 15]. It's an evolutionary drive consequence [11]. We want to stress that, seldom have these practical restrictions on topology been considered in previous investigations.

Local attachment also suggests lacking of shortcuts, the diameter  $D_0$  (largest distance between nodes) of resulting scale-free-fractal network in turn scales as  $N^{1/d_B}$  ( $d_B$  is fractal dimension, see Eq. (2)). Longer-range entangled links are successively created by performing QRP, constituting larger-scale QRN. *It's equal to implement certain transformation on the structure of QRN assisted by QRP.* To be definite, an organizing principle shaping the topology is required to control the allocation and layout of long-range entanglements across network. A realistic design of QRN should satisfy above requirements. One may easily think of randomized distribution of shortcuts. Yet it's not an option in quantum setting. Apparently, it's unable to keep neither the hierarchy of measurements nor scale-free property of large-scale QRN. Instead, we harness renormalization to control the distribution of shortcuts.

We relate the hierarchical trait of measurements to the coarse-graining process, and map the distribution of shortcuts to renormalization transformation. We find that our scenario could be seen as a natural extension of 1D case, and there exist subtle corresponding relationships between them. Compared with segmentation fashion of 1D quantum repeater chain, the whole network is divided into boxes, whose size is equal to the number of segments of each unit (see Fig. 1). Meanwhile, we get a general expression of nesting level [3] which increases logarithmically with size of network (see Eq. (8)). The highlights of this scenario are as follows. The distribution of shortcuts becomes a process of collectively implementing QRP across the entire network, where the hierarchy of measurement is preserved at network level, while the scale-free-fractal structure is kept at all length-scales. And large-scale scale-free-fractal QRN are built via consecutive enlargement with some length-scales. Besides, we will both analytically and numerically prove that small-world is obtained.

Renormalization is a powerful tool for both quantum field theory and phase transition in statistical physics. It was successfully introduced into complex networks by Song *et al.*, uncovering self-similarity of complex networks [12]. A network with total number of nodes  $N$  is renormalized [13] according to the box-covering technique [12] (see Fig. 1(c)): tile the entire network with minimum number of boxes  $N_B$ , where the distance between nodes within any box is smaller than the box size, namely, the transforming length-scale  $\ell_B$ . Each box is then replaced by a supernode. These supernodes are connected if there is at least one link between nodes in their respective boxes. This method defines the fractal dimension  $d_B$  in terms of a power law:

$$\frac{N_B}{N} \sim \ell_B^{-d_B}. \quad (2)$$

Apply this transformation  $R_{\ell_B}$  to a fractal network  $G_0$ , then we have  $R_{\ell_B}(G_0) = G_0$ , which is scale invariant.

Several algorithms have been proposed to coarse-grain

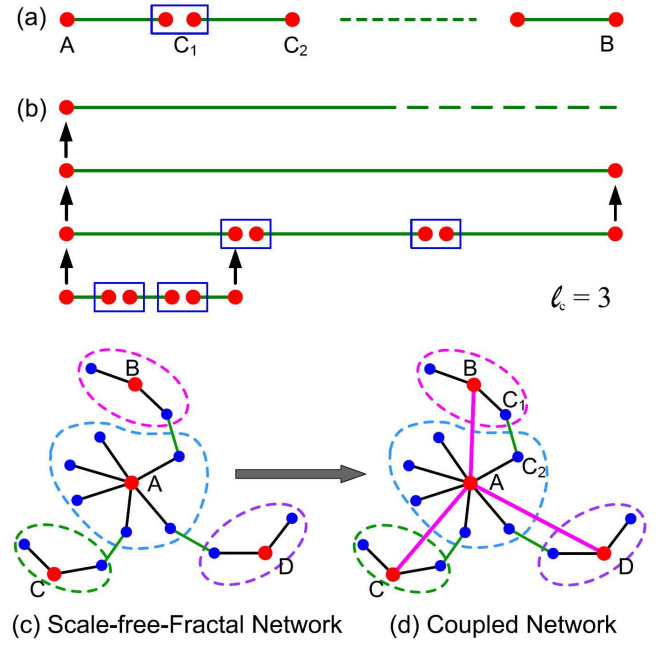


FIG. 1: (color online). Renormalization and its relationship with quantum repeaters. Top two panels: principle of quantum repeaters. (a) Entanglements are repeatedly created for each auxiliary segment of the channel, the length of which is shorter than the attenuation length. (b) Nested purification that combines entanglement swapping and purification are successively performed in a hierarchical way, which actually corresponds to coarse-graining process in network setting, as exhibited in Fig.1(d). Adjacent segments are connected and extended to longer distance, eventually two remote nodes are connected via perfect entanglements. Bottom panel: schematic illustration of coupled renormalization. (c) Nodes are assigned to different boxes according to the MEMB algorithm. (d) The entangled link lying between two boxes (green links) makes a short path whose length is about  $\ell_c$  connecting respective hubs. And we return to 1D case, e.g., hubs A and B are connected via a path  $B-C_1-C_2-A$ . Then longer-range entanglements are established between them (pink links) by performing QRP.

complex networks, nevertheless, not all of them are useful here. With the advantages of guaranteeing connectivity within boxes, isolating hubs of different boxes and avoiding overlap between boxes, we choose the MEMB algorithm [14].

In order to utilize renormalization as a strategy of distributing shortcuts, some improvements are necessary. We select the hub of a box (node with largest degree) as representative node, representing the box. If there is one link between two boxes, place one shortcut connecting their hubs. Otherwise, no links are attached. As shown in Fig.1(d), pink links denote corresponding shortcuts, constituting the coarse-grained network (CGN). As a result, CGN is reconstructed and coupled to the initial network. Until now, we have three networks: the initial, the coarse-grained, and the superposition of the two, i.e., the coupled network (CN). For the sake of distinction between the normal and this modified renormalization, we name the later *coupled renormalization* (CR).

Above presentation does not involves any physical context, but instructs us to create shortcuts between which pairs of nodes. Without loss of generality, we apply CR to a scale-free-fractal network generated by the minimal model [11]. See supplemental material. In geographic space, nodes are initially connected by elementary links according to above network configuration. And entanglements are repeatedly created for each elementary link. Ref. [16] presented a mechanism with multiplexing setup allowing near-deterministic creation of entanglement between adjacent quantum nodes. The creation of entanglements between two qubits are core building blocks of quantum repeaters. The seminal protocol of DLCZ has stimulated substantial theoretical and experimental works toward realization of QRP based on atomic ensembles and linear optics [2, 3, 5]. Here our scenario is a framework that is not restricted to certain special QRP, but compatible with various QRP. To obtain quantum small-world, CR is performed with transforming length-scale  $\ell_c$ , e.g., as shown in Fig. 1 (b) and (d),  $\ell_c = 3$ . Notice that we don't create shortcut between two hubs if there is no link between their boxes (e.g, hubs C and D). In such circumstance, the distance between them is much bigger than  $\ell_c$ .

Owing to the self-similarity of complex networks, the resulting CGN is topologically equivalent to the original one. What's more, the initial network is enlarged into CGN with length-scale  $\ell_c$  in space. Once it is completed, next round of CR can be iteratively applied to previous CGN with fixed  $\ell_c$ , and so on until the critical nesting level (see Eq. (8)) is reached. Let  $G_n$  be the  $n$ th CGN. According to renormalization group theory,

$$G_n = R_{\ell_c}(G_{n-1}) = R_{\ell_c}^n(G_0) = R_{\ell_c}^n(G_0), \quad (3)$$

so  $G_n$  equivalently results from single CR with transforming length-scale  $\ell_c^n$ . Larger-scale CGN here act as shortcuts of the underlying smaller-scale ones entangling nodes separated by long distance, which dramatically changes the topology in such a way that nodes are globally separated by short chains of entangled links. This can be further revealed by the distance distribution which follows Gaussian distribution (see Fig. 2(b)). Hence, in the end, a hierarchical quantum network with small-world property is produced.

It's crucial to make it clear whether single or iterative CR will lead to fractal to small-world transition. Two analytical proofs with numerical simulations are provided. A rigorous and reliable method is to observe the behavior of average degree under renormalization flow [15]. Detailed proof is included in supplemental material. In regard to single CR, the expected transition doesn't occur. However, it's safe to say that iterative CR can give rise to fractal to small-world transition. Evidences for the transition displayed in Fig. 3 conform above conclusion. Nonetheless, the average diameter of CN, namely, the signature of small-world networks is unclear.

To evaluate the diameter of iterative CN,  $D(n, \ell_c)$ , we begin with analyzing the impact of single CR, and then generalize it to iterative CR case. We devise a hierarchical routing method

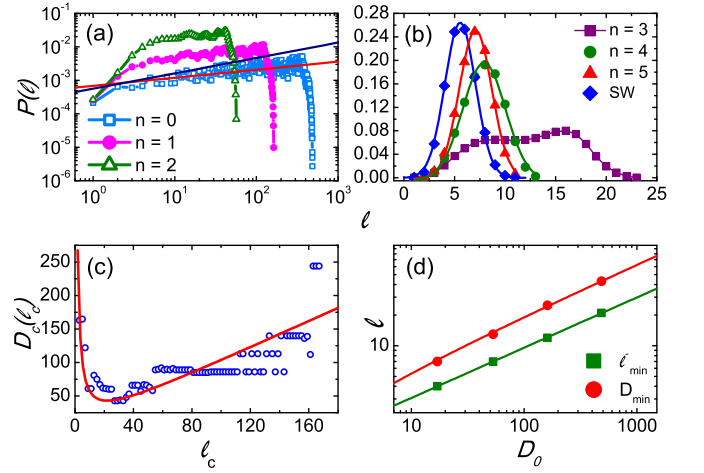


FIG. 2: (color online). Statistical properties of CN. In this example,  $t = 5$ ,  $N = 9375$ ,  $\ell_c = 3$ . (a) and (b) Distance distribution of iterative CN. (a) is log-log plot of  $P(\ell)$  versus  $\ell$ , the slope of the upper line is  $d_B - 1 \approx 0.46$  (analytical estimate), and the lower line 0.24 (fitting). (c) Prediction of diameter of single CN. (d) Log-log plot of the minimal and average diameter of different size of single CN.

and obtain  $D_C(\ell_c)$ , the diameter of CN:

$$D_C(\ell_c) \approx \frac{D_0}{\ell_c} + \ell_c - 1, \quad (4)$$

see Fig. 2(c). Take note that there is an optimal transforming length-scale which yields minimal diameter  $D_{\min}(N) = 2\sqrt{D_0} - 1$ . Whereas the corresponding minimal average diameter,  $\bar{\ell}_{\min}(N) = D_{\min}(N)/2 \approx \sqrt{D_0}$  (see Fig. 2(d)), suggesting that single CR is unable to trigger the transition, which is consistent with above conclusion. See supplemental material for derivation and interpretation.

Now let's consider iterative CR with fixed box size. By analogy with above results, it's easy to obtain the diameter of iterative CN for small  $\ell_c$ , using Eq. (4) by recursive derivation, we have

$$D(n, \ell_c) \approx \frac{D_0}{\ell_c^n} + n(\ell_c - 1), \quad (5)$$

where  $n$  is called nesting level, namely, the times of iterative CR. Taking into account finite size effect, Eq. (5) holds on condition that  $\ell_c \ll \sqrt{D_0}$ .

In particular, the first term decays exponentially, whereas the second increases linearly. Hence,  $D(n, \ell_c)$  is governed by the linear term and grows slowly. We readily obtain the criterion for the transition:

$$\ell_c^{n_c} \sim D_0, \quad (6)$$

implying that  $D(n_c, \ell_c) \approx n_c(\ell_c - 1)$  for large  $N$ . It's desirable that both  $n_c$  and  $D(n_c, \ell_c)$  increases logarithmically with size of network, since

$$D(n_c, \ell_c) \approx \frac{\ell_c - 1}{d_B \ln \ell_c} \ln N, \quad (7)$$

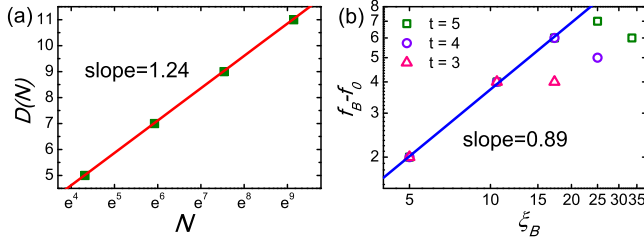


FIG. 3: (color online). Evidences for small-world networks. (a) Logarithmical increasing of diameter for iterative CN with fixed  $\ell_c = 3$ . (b) Log-log plot of  $f_B - f_0$  versus  $\xi_B$ .

and

$$n_c \sim \frac{\ln N}{d_B \ln \ell_c}. \quad (8)$$

Here  $n_c$ , which we call *critical nesting level*, is inversely proportional to fractal dimension  $d_B$ , explicitly indicating its topological interdependence. When  $d_B = 1$ , Eq. (8) reproduces the result of 1D case. Eq. (7) tells us that, once again, small-world is obtained. What's more, it's exactly in agreement with simulation result, see Fig. 3(a).

Obviously, CR can be applied to other type of networks as well, such as square lattice, triangular lattice and random networks (ER) at criticality [12], etc. However, such topological structure is invalid from a practical point of view. The phenomenon discussed in Ref. [6] is striking and impressive, though, random graph configuration is not a realistic option.

In contrast, we provide a paradigmatic quantum network which is scalable and open to the growth of network. Long-range entanglement is created by performing QRP, with which the exponential decay of communication fidelity could be mitigated and overcome in very long noisy channels. Our scenario is a framework applicable to various QRP including future mature schemes. Moreover, both the critical nesting level and diameter increase logarithmically with size of networks. Additionally, thanks to the scale-free nature, CN is particularly robust to random failure of entangled links [7, 17].

In summary, we have analyzed the basic rules that the topology of quantum networks should obey, and presented a framework of building quantum small-world networks based on renormalization and the principle of quantum repeaters. Be-

yond that, we draw on concepts and methodology from statistical physics which prove inspiring for studying quantum networks. Finally, the crossover investigation which is still in its infancy, has shifted the perspective of quantum information theory from simple small system to complex system. Thereby more fresh ideas and intriguing phenomena are expected.

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